2015 Mat-2 Final

1) $\vec{r}(t) = t\vec{i} + t^2\vec{j} + (t^2-1)\vec{k}$ eğrisinin $t=1$ anındaki eğriliğini bulunuz. (10p)

$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 2t\vec{k}$, $\vec{r}\,''(t) = 2\vec{j} + 2\vec{k}$

$t=1$ için

$\vec{r}\,'(1) = \vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{r}\,''(1) = 2\vec{j} + 2\vec{k}$

$$K(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \Rightarrow \frac{\left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{matrix} \right\|}{\left(\sqrt{1^2+2^2+2^2}\right)^3} = \frac{\sqrt{8}}{27}$$

2) $A = \begin{bmatrix} 0 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ matrisinin özvektörlerini bul (15p)

$$[A - \lambda I] = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 & 3 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(1-\lambda)^2 = 0 \quad \lambda_1 = 0 \quad \lambda_2 = \lambda_3 = 1$$

$\lambda_1 = 0$ için

$$\begin{bmatrix} 0 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$-x_1 + x_2 = 0 \quad x_2 = x_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_1 \quad x_1 \in \mathbb{R} - \{0\}$$

$\lambda_2 = \lambda_3 = 1$

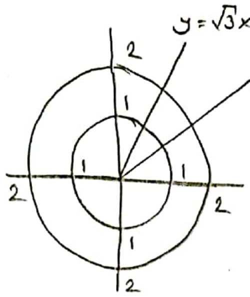
$$\begin{bmatrix} -1 & 0 & 3 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 3x_3 = 0 \quad x_1 = -3x_3$

$x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 \quad x_2 \in \mathbb{R} - \{0\}$$

3) Birinci bölgede $x^2+y^2=1$, $x^2+y^2=4$, $y=x$, $y=\sqrt{3}x$ ile sınırlı bölgenin alanını iki katlı integralle hesaplayınız. (15p)



$y=\sqrt{3}x \Rightarrow \theta=\frac{\pi}{3}$

$y=x \rightarrow \theta=\frac{\pi}{4}=45°$

$x^2+y^2=1$ $\quad x=0 \Rightarrow y=\mp 1$ $\quad y=0 \Rightarrow x=\mp 1$

$x^2+y^2=4$ $\quad x=0 \Rightarrow y=\mp 2$ $\quad y=0 \Rightarrow x=\mp 2$

$u=x^2+y^2$, $1\leqslant u\leqslant 4$

$v=\frac{y}{x}$, $1\leqslant v\leqslant \sqrt{3}$

$$\mathcal{J}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} 2x & 2y \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix}=2+\frac{2y^2}{x^2}=2+2v^2$$

$$J=\frac{1}{\mathcal{J}}=\frac{1}{2(1+v^2)} \quad , \quad A=\iint_B |J|\,dA=\frac{1}{2}\int_{u=1}^{4}\left(\int_{v=1}^{\sqrt{3}}\left(\frac{dv}{1+v^2}\right)du\right.$$

$$A=\frac{1}{2}\int_1^4\left(\int_1^{\sqrt{3}}\frac{dv}{v^2+1}\right)du=\frac{1}{2}\int_1^4 \arctan v\Big|_1^{\sqrt{3}}\,dv$$

$$A=\frac{1}{2}\int_1^4(\underbrace{\arctan\sqrt{3}}_{\frac{\pi}{3}}-\underbrace{\arctan 1}_{\frac{\pi}{4}})\,du=\frac{1}{2}\frac{\pi}{12}\int_1^4 du \qquad A=\frac{\pi}{24}u\Big|_1^4=\frac{\pi}{8}\ br^2$$

4) $z=\sqrt{x^2+y^2}$ ile $z=0$, $z=3$ arasında kalan bölgenin hacmini hesaplayınız.

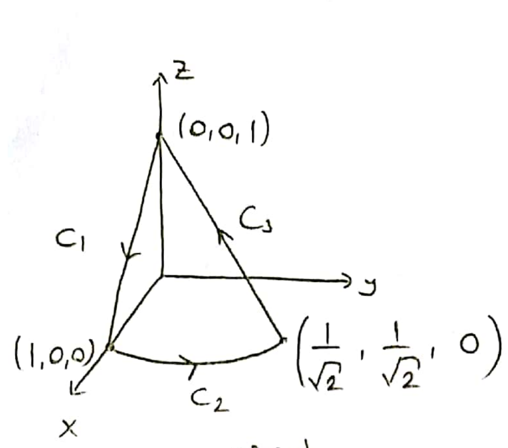
$$V=\iiint_{(G)}dV=\iint_{(B)}\left(\int_{z=\sqrt{x^2+y^2}}^{3}dz\right)dA=\iint_{(B)}(z)\Big|_{z=\sqrt{x^2+y^2}}^{3}dA=\iint_{(B)}\left[3-\sqrt{x^2+y^2}\right]dA$$

$$\boxed{\begin{array}{l} x=r.\cos\theta \\ y=r.\sin\theta \\ 0\leqslant\theta\leqslant 2\pi \\ 0\leqslant r\leqslant 3 \end{array}}$$

$$\int_0^{2\pi}\left(\int_0^3(3-r)r\,dr\right)d\theta=\int_0^{2\pi}\left(\frac{3}{2}r^2-\frac{r^3}{3}\Big|_0^3\right)d\theta=\int_0^{2\pi}\left(\frac{27}{2}-3\right)d\theta$$

$$\Rightarrow A=\frac{9}{2}\theta\Big|_0^{2\pi}=9\pi\ br^3$$

5)



sekle göre $I = \int\limits_{C_1+C_2+C_3} (y\,dx - x\,dy + dz) = ?$ (15p)

$C_1: (0,0,1) \xrightarrow{\text{yönünde}} (1,0,0)$ doğru

$C_1: \frac{x}{1} = \frac{y}{0} = \frac{z-1}{-1} = t$ den $\qquad C_1: \begin{array}{l} x = t \\ y = 0 \\ z = 1-t \end{array} \qquad 0 \leq t \leq 1$

$C_2: x^2+y^2=1 \rightarrow C_2: \begin{array}{l} x = \cos t \\ y = \sin t \\ z = 0 \end{array}, \quad 0 \leq t \leq \frac{\pi}{4}$

$C_3: \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \xrightarrow{\text{yönünde}} (0,0,1)$

$\frac{x-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = \frac{y-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = \frac{z}{1} = t \qquad C_3: \begin{cases} x = \frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}, \quad 0 \leq t \leq 1 \\ z = t \end{cases}$

$$I = \int\limits_{C_1+C_2+C_3} (y\,dx - x\,dy + dz) + \int_0^{\frac{\pi}{4}} \left(\frac{\sin t(-\sin t)dt - \cos t . \cos t\, dt + z}{dt}\right)$$

$$+ \int_0^1 \left[\frac{\left(\frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}\right)\left(-\frac{dt}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}\right)\left(-\frac{dt}{\sqrt{2}}\right) + dt}{dt}\right]$$

$$I = -t\Big|_0^1 - t\Big|_0^{\pi/4} + t\Big|_0^1$$

$$I = -1 - \frac{\pi}{4} + 1 = \underline{\underline{-\frac{\pi}{4}}}$$