Mat 2 2016 Final

1)

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 5 & 3 & 0 \\ 6 & 0 & -4 \end{bmatrix}$$ matrisinin özdeğerlerini ve özvektörlerini bulunuz. (20p)

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & 0 & -2 \\ 5 & 3-\lambda & 0 \\ 6 & 0 & -4-\lambda \end{bmatrix}$$

$= (4-\lambda).(3-\lambda).(-4-\lambda) + 2.6.(3-\lambda)$

$= (3-\lambda)(-16+\lambda^2+12) = 3-\lambda(\lambda^2-4) = (3-\lambda).(\lambda^2-2^2)$

$= (3-\lambda).(\lambda-2).(\lambda+2)$

özdeğerler: $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 3$

$\lambda_1 = -2$ için

$$\begin{bmatrix} 6 & 0 & -2 \\ 5 & 5 & 0 \\ 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 6x_1 - 2x_3 = 0 \\ 5x_1 + 5x_2 = 0 \end{matrix}\right\} \begin{matrix} x_3 = 3x_1 \\ x_2 = -x_1 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 3x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} x_1, \quad x_1 \in \mathbb{R} - \{0\}$$

$\lambda_2 = 2$ için

$$\begin{bmatrix} 2 & 0 & -2 \\ 5 & 1 & 0 \\ 6 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 2x_1 - 2x_3 = 0 \\ 5x_1 + x_2 = 0 \end{matrix}\right\} \begin{matrix} x_3 = x_1 \\ x_2 = -5x_1 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -5x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} x_1, \quad x_1 \in \mathbb{R} - \{0\}$$

$\lambda_3 = 3$ için

$$\begin{bmatrix} 1 & 0 & -2 \\ 5 & 0 & 0 \\ 6 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} x_1 - 2x_3 = 0 \\ 5x_1 = 0 \\ 6x_1 - 7x_3 = 0 \end{matrix}\right\} \begin{matrix} x_1 = 2x_3 \Rightarrow x_3 = 0 \\ x_1 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2, \quad x_2 \in \mathbb{R} - \{0\}$$

2) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ fonksiyonunun ekstremumlarını inceleyiniz. (15p)

$f_x(x,y) = 3x^2 + 3y^2 - 6x = 0$

$f_y(x,y) = 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0$

$x = 1 \lor y = 0$

$x = 1$ için $3 + 3y^2 - 6 = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y = \mp 1$

$y = 0$ için $3x^2 - 6x = 0 \Rightarrow x(3x - 6) = 0 \Rightarrow x = 0 \lor x = 2$

kritik noktalar $(1,-1)$, $(1,1)$, $(0,0)$, $(2,0)$

$f_{xx} = 6x - 6$, $f_{yy} = 6x - 6$, $f_{xy} = 6y$

$(1,-1)$ için

$f_{xx}(1,-1)\, f_{yy}(1,-1) - f_{xy}^2(1,-1) < 0$ olduğundan f fonksiyonu $(1,-1)$'de bir eyer noktasına sahiptir.

$(1,1)$

$f_{xx}(1,1)\, f_{yy}(1,1) - f_{xy}^2(1,1) < 0$ old. f fonk. $(1,1)$'de bir eyer nok. sahip.

$(0,0)$

$f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) > 0$ ve $f_{xx}(0,0) < 0$ old. f fonk. $(0,0)$ da bir yerel max. sahip.

$(2,0)$ için

$f_{xx}(2,0) \cdot f_{yy}(2,0) - f_{xy}^2(2,0) > 0$ ve $f_{xx} > 0$ old. f fonksiyonu $(2,0)$'da bir yerel min. sahip.

$6 \cdot 6 -$

3) $x + 3y - 2z = 4$ düzleminin orijine olan uzaklığını Lagrance çarpanları kuralını kullanarak bul. (15p)

$f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x + 3y - 2z - 4 = 0$

$\nabla f = \lambda \nabla g \Rightarrow 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} + 3\vec{j} - 2\vec{k})$ $\quad \lambda\vec{i} + \lambda 3\vec{j} - \lambda 2\vec{k}$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= 3\lambda \\ 2z &= -2\lambda \\ x + 3y - 2z &= 4 \end{aligned}\right\} \quad \frac{\lambda}{2} + 3 \cdot \frac{3}{2}\lambda - 2\left(-\frac{2\lambda}{2}\right) = 4$$

$14\lambda = 8 \Rightarrow \lambda = \frac{8}{14} = \frac{4}{7}$

$(x,y,z) = \left(\frac{4}{14}, \frac{12}{14}, -\frac{4}{7}\right)$

$x + 3y - 2z = 4$ düzleminin orijine olan uzaklığı

$$d = \sqrt{\left(\frac{4}{14}\right)^2 + \left(\frac{12}{14}\right)^2 + \left(-\frac{4}{7}\right)^2} = \sqrt{\frac{16}{196} + \frac{144}{196} + \frac{16}{49}} = \sqrt{\frac{16}{196} + \frac{144}{196} + \frac{64}{196}} = \sqrt{\frac{224}{196}} = \sqrt{\frac{8}{7}}$$

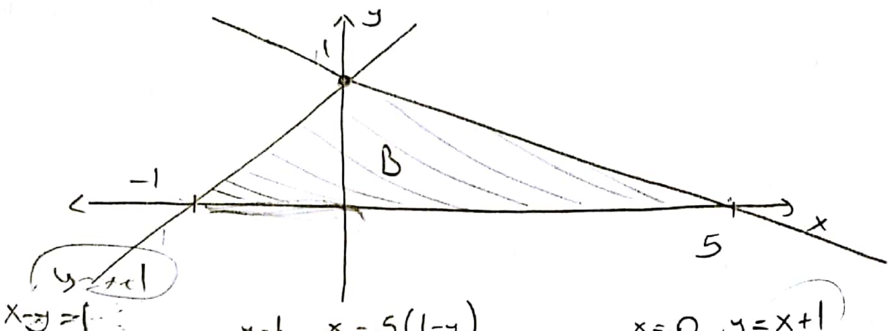
(4)

$= \frac{4}{\sqrt{14}}$

4) B integrasyon bölgesi, $x-y=-1$, $x+5y=5$ ve x ekseni ile sınırlı bölge olsun. Buna göre $\iint_B x\,dx\,dy$ integralini hesaplayınız. (15p)

$x-y=-1$; $x=0 \Rightarrow y=1$, $y=0 \Rightarrow x=-1$

$x+5y=5$; $x=0 \Rightarrow y=1$, $y=0 \Rightarrow x=5$



$\frac{1}{5}x+y=1$

$$\iint_B x\,dx = \int_{y=0}^{y=1}\int_{x=y-1}^{x=5(1-y)} x\,dx\,dy = \int_{x=-1}^{x=0}\int_{y=0}^{y=x+1} x\,dy\,dx + \int_{x=0}^{x=5}\int_{y=0}^{y=1-\frac{1}{5}x} x\,dy\,dx$$

$$= -\frac{1}{6}+\frac{5}{2}=\frac{14}{6}=\frac{7}{2} \;? \quad = \frac{-1}{6}$$

5) $z=6-x^2-y^2$ paraboloidi ve $z=-3$ düzlemi ile sınırlı bölgenin hacmini bulunuz (15p)

$-3=6-x^2-y^2 \Rightarrow x^2+y^2=9 \;\; 3^2$ $\qquad r=3$

$$V=\iint_B 6-x^2-y^2-(-3)\,dx\,dy=\iint_B 9-(x^2+y^2)\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^{3}(9-r^2)\,r\,dr\,d\theta=\int_0^{2\pi}\frac{9}{2}r^2-\frac{r^4}{4}\Big|_0^3\,d\theta=\int_0^{2\pi}\frac{81}{4}\,d\theta=\frac{81}{2}\pi$$

$$\frac{81}{2}-\frac{81}{4}=\frac{81}{4}$$

6) $z^2 = x^2+y^2$ ile $z=2$ nin arakesit yüzeyinin yüzey alanını bulunuz. (15p)

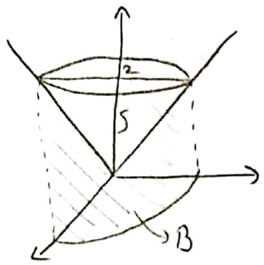
$z = \sqrt{x^2+y^2}$, $x^2+y^2=4$

$x = r.\cos\theta$
$y = r.\sin\theta$
$0 \leq \theta \leq 2\pi$
$0 \leq r \leq 2$

$$z_x = \frac{x}{\sqrt{x^2+y^2}} \qquad z_y = \frac{y}{\sqrt{x^2+y^2}}$$

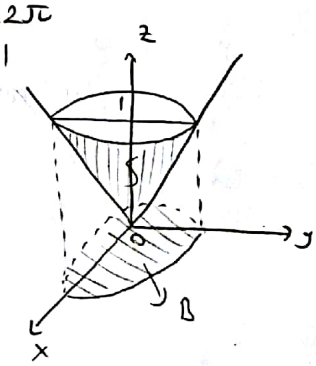
$$S = \iint_B \sqrt{1+z_x^2+z_y^2}\, dA = \iint_B \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\, dA$$

$$S = \iint_B \sqrt{2}\, dA = \sqrt{2}\int_0^{2\pi}\left(\int_0^2 r\,dr\right)d\theta = \sqrt{2}\int_0^{2\pi} \frac{r^2}{2}\Big|_0^2 d\theta = 2\sqrt{2}\int_0^{2\pi} d\theta \Rightarrow S = 4\sqrt{2}\pi \text{ br}^2$$



7) $\vec{F} = z\vec{i} + \vec{j} + \vec{k}$, $S: z=\sqrt{x^2+y^2}$ nin $z=0$, $z=1$ ile ayrılan parçası olmak üzere ($\vec{n}$ içe yönlü) $I = \iint_{(S)} \text{rot}\,\vec{F}.\vec{n}.ds = ?$ (15p)

$x = r.\cos\theta$
$y = r.\sin\theta$
$0 \leq \theta \leq 2\pi$
$0 \leq r \leq 1$



$$\vec{n} = \frac{-z_x\vec{i} - z_y\vec{j} + \vec{k}}{\sqrt{1+z_x^2+z_y^2}}$$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 1 & 1 \end{vmatrix} = \vec{j}, \quad z_x = \frac{x}{\sqrt{x^2+y^2}} \quad z_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{rot}\,\vec{F}.\vec{n} = \vec{j}.\vec{n} = \frac{-z_y}{\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}} = -\frac{z_y}{\sqrt{2}}$$

$$I = \iint_S \text{rot}\,\vec{F}.\vec{n}.ds = -\frac{1}{\sqrt{2}}\iint_S \frac{y\,ds}{\sqrt{x^2+y^2}}$$

$$= -\frac{1}{\sqrt{2}}\iint_B \frac{y}{\sqrt{x^2+y^2}}\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\, dA = -\frac{1}{\sqrt{2}}\iint_B \sqrt{2}\frac{y\,dA}{\sqrt{x^2+y^2}}$$

$$I = -\int_0^{2\pi}\left(\int_0^1 \frac{r\sin\theta}{r}.r\,dr\right)d\theta = -\int_0^{2\pi}\sin\theta \frac{r^2}{2}\Big|_0^1 d\theta = -\frac{1}{2}\int_0^{2\pi}\sin\theta\, d\theta$$

$$I = \frac{1}{2}\cos\theta\Big|_0^{2\pi} = \frac{1}{2}(\underset{1}{\cos 2\pi} - \underset{1}{\cos 0}) = \frac{1}{2}.0 = \underline{\underline{0}}$$